

# The Maslov Indices of Hamiltonian Periodic Orbits

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## Abstract

We use the properties of the Leray index to give precise formulas in arbitrary dimensions for the Maslov index of the monodromy matrix arising in periodic Hamiltonian systems. We compare our index with other indices appearing in the literature.

## 1 Introduction

There have been in the recent years many papers discussing the “Maslov index” of periodic Hamiltonian orbits; the papers by Brack and Jain [1], Creagh and Littlejohn [2], Robbins [11], and Sugita [12] are certainly major advances. One of the main reasons of the interest in the topic comes from the fact that these indices play a crucial role in the determination of the correct phases in Gutzwiller’s trace formula [8], which is one of the main tools in the study the spectrum of semiclassical systems associated to non-integrable Hamiltonians.

The purpose of this Letter is to relate that Maslov index to another index, of cohomological nature, defined by Leray [9] in the transversal case, and extended by the author [4, 5] to the general case, and which has been used in [6] in another context. Our approach will highlight the role played by the index of inertia of triples of Lagrangian planes, which can be viewed as a Morse index, and completes the Note Added in Proof in the recent paper [10] by Pletyukhov and Brack.

## 2 Leray and Maslov Indices

We shortly review the properties of the Leray and Maslov indices that we will need; for proofs and details see [4, 5]. Let  $\sigma(z, z') = p \cdot x' - p' \cdot x$  be the standard

symplectic form on  $\mathbb{R}_z^{2n} \equiv \mathbb{R}_x^n \times \mathbb{R}_p^n$  ( $z = (x, p)$ ,  $z' = (x', p')$ ), that is

$$\sigma(z, z') = Jz \cdot z' = (z')^T Jz \quad \text{with} \quad J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix}.$$

The corresponding symplectic group is denoted by  $Sp(n)$ ; the unitary group  $U(n, \mathbb{C})$  is identified with a compact subgroup  $U(n)$  of  $Sp(n)$  by the isomorphism

$$A + iB \mapsto \begin{pmatrix} A & -B \\ B & A \end{pmatrix}.$$

The Lagrangian Grassmannian  $\Lambda(n)$  consists of all  $n$ -dimensional subspaces  $\ell$  of  $\mathbb{R}_z^{2n}$  on which  $\sigma$  vanishes identically;  $U(n)$  (and hence  $Sp(n)$ ) acts transitively on  $\Lambda(n)$ . Let  $\ell_p = 0 \times \mathbb{R}_p^n$  be the momentum plane; the ‘‘Souriau mapping’’

$$\ell = u\ell_p \mapsto uu^T \quad (1)$$

identifies  $\Lambda(n)$  with the manifold  $W(n, \mathbb{C}) \subset U(n, \mathbb{C})$  of all *symmetric* unitary matrices; the action of  $U(n, \mathbb{C})$  on  $\Lambda(n) \equiv W(n, \mathbb{C})$  is then defined by

$$u\ell = u w u^T \quad \text{if} \quad \ell \equiv w.$$

A useful result is that if  $w \equiv \ell$  and  $w' \equiv \ell'$ , then

$$\text{rank}(w - w') = n - \dim \ell \cap \ell'. \quad (2)$$

The universal coverings of  $Sp(n)$ ,  $U(n, \mathbb{C})$  and  $\Lambda(n) \equiv W(n, \mathbb{C})$  are denoted by  $Sp_\infty(n)$ ,  $U_\infty(n, \mathbb{C})$  and  $\Lambda_\infty(n) \equiv W_\infty(n, \mathbb{C})$ . Recall that  $Sp_\infty(n)$  is (as is  $U_\infty(n, \mathbb{C})$ ) a group; the composition law is as follows: if  $S_\infty$  and  $S'_\infty$  are the homotopy equivalence classes of two symplectic paths  $t \mapsto S_t$ ,  $t \mapsto S'_t$ ,  $0 \leq t \leq a$  originating at the identity, then  $S_\infty S'_\infty$  is the homotopy class of the path  $t \mapsto S_t S'_t$ . We have the following identifications (see e.g. [7]):

$$U_\infty(n, \mathbb{C}) = \{(u, \theta) : u \in U(n, \mathbb{C}), \det u = e^{i\theta}\}$$

and

$$W_\infty(n, \mathbb{C}) = \{(w, \theta) : w \in W(n, \mathbb{C}), \det w = e^{i\theta}\}.$$

The Leray index of  $(\ell_\infty, \ell'_\infty) \equiv (w, \theta; w', \theta')$  is defined as follows: (\*) if  $\ell \cap \ell' = 0$ , then  $m$  is given by the formula

$$m(\ell_\infty, \ell'_\infty) = \frac{1}{2\pi}(\theta - \theta' + i \text{Tr} \text{Log}(-w(w')^{-1})) + \frac{n}{2}. \quad (3)$$

Notice that (2) implies that  $\det(I - w(w')^{-1}) \neq 0$ , hence  $-w(w')^{-1}$  has no eigenvalues on the negative half-line. This allows us to choose for Log the usual logarithm; (\*\*) if  $\ell \cap \ell' \neq 0$  then choose  $\ell''_\infty$  such that  $\ell \cap \ell'' = \ell' \cap \ell'' = 0$  and set

$$m(\ell_\infty, \ell'_\infty) = m(\ell_\infty, \ell''_\infty) - m(\ell'_\infty, \ell''_\infty) + \text{Inert}(\ell, \ell', \ell'') \quad (4)$$

where the “index of inertia” of the triple  $(\ell, \ell', \ell'')$  is defined by:

$$\text{Inert}(\ell, \ell', \ell'') = \frac{1}{2}(\tau(\ell, \ell', \ell'') - \partial \dim(\ell, \ell', \ell'') + n). \quad (5)$$

We will come back to definition (4) in a moment; let us first make a short detour to explain (5); for details we refer to [4, 5]. First

$$\partial \dim(\ell, \ell', \ell'') = \dim(\ell \cap \ell') - \dim(\ell \cap \ell'') + \dim(\ell' \cap \ell'')$$

is the “coboundary” of the “cochain”  $\dim(\ell, \ell') \equiv \dim(\ell \cap \ell')$ ; as we will see below it makes  $\text{Inert}(\ell, \ell', \ell'')$  become an integer. The term  $\tau(\ell, \ell', \ell'')$  is called the “signature” (or “Kashiwara index”) of the triple  $(\ell, \ell', \ell'')$ . It is the difference  $\tau^+ - \tau^-$  between the number of  $> 0$  and  $< 0$  eigenvalues of the quadratic form  $\sigma(z, z') + \sigma(z', z'') + \sigma(z'', z)$  on  $\ell \times \ell' \times \ell''$ . By definition of  $Sp(n)$  as the group of linear automorphisms leaving  $\sigma$  invariant it immediately follows that  $\tau(\ell, \ell', \ell'')$  is a symplectic invariant, that is,  $\tau(S\ell, S\ell', S\ell'') = \tau(\ell, \ell', \ell'')$  for every  $S \in Sp(n)$ . Since the symplectic form  $\sigma$  is antisymmetric, the signature changes sign when one swaps any two of the planes in its argument: for instance  $\tau(\ell', \ell, \ell'') = -\tau(\ell, \ell', \ell'')$ ,  $\tau(\ell'', \ell', \ell) = -\tau(\ell, \ell', \ell'')$ , and so on. In particular  $\tau(\ell, \ell', \ell'') = 0$  if any two of the planes  $\ell, \ell', \ell''$  are identical. A much less trivial property of the signature is the following: it is a “cocycle”, in the sense that  $\partial\tau = 0$ ; explicitly:

$$\tau(\ell_1, \ell_2, \ell_3) - \tau(\ell_2, \ell_3, \ell_4) + \tau(\ell_1, \ell_3, \ell_4) - \tau(\ell_1, \ell_2, \ell_4) = 0$$

for all 4-tuplets  $(\ell_1, \ell_2, \ell_3, \ell_4)$ ; it is proven using elementary (but lengthy) arguments of linear algebra (see [5]). Finally, one shows by studying the kernel of the bilinear form associated with  $\sigma(z, z') + \sigma(z', z'') + \sigma(z'', z)$  that

$$\tau(\ell, \ell', \ell'') \equiv n + \partial \dim(\ell, \ell', \ell'') \pmod{2};$$

this implies that the index of inertia indeed is an integer as claimed. The following properties of  $\text{Inert}(\ell, \ell', \ell'')$  immediately follow from those of the signature: it is a symplectic invariant in the sense that

$$\text{Inert}(S\ell, S\ell', S\ell'') = \text{Inert}(\ell, \ell', \ell'') \quad (6)$$

for every  $S \in Sp(n)$ ; it is a cocycle:  $\partial \text{Inert} = 0$ , that is

$$\begin{aligned} & \text{Inert}(\ell_1, \ell_2, \ell_3) - \text{Inert}(\ell_2, \ell_3, \ell_4) + \text{Inert}(\ell_1, \ell_3, \ell_4) \\ & - \text{Inert}(\ell_1, \ell_2, \ell_4) = 0. \end{aligned} \quad (7)$$

From the antisymmetry of the signature follows that we have the relations

$$\begin{aligned} \text{Inert}(\ell, \ell, \ell'') &= \text{Inert}(\ell, \ell', \ell') = 0 \\ \text{Inert}(\ell, \ell', \ell) &= n - \dim \ell \cap \ell' \end{aligned}$$

(observe that  $\text{Inert}$  does not inherit the antisymmetry property of  $\tau$ ; this is due to the presence of the correcting term  $\partial \dim$  in its definition).

Both the signature and the index of inertia are symplectic invariant measures of the relative positions of Lagrangian planes; let us illustrate this the simple case  $n = 1$ :

**Example 1** *When  $n = 1$  the Lagrangian Grassmannian is the set of all lines through the origin in the phase plane. If the line  $\ell'$  lies inside the open angular sector delimited by  $\ell$  and  $\ell''$  (we assume these lines oriented) then  $\tau(\ell, \ell', \ell'') = -1$ ; if it lies outside that sector then  $\tau(\ell, \ell', \ell'') = +1$ ; the signature is zero in the other cases. It follows that  $\text{Inert}(\ell, \ell', \ell'')$  is 0 if  $\ell'$  is in the closed angular sector delimited by  $\ell$  and  $\ell''$ , and  $\pm 1$  otherwise.*

Let us now return to the definition (4) of the Leray index. That the right-hand side of (4) does not depend on the choice of  $\ell''_\infty$  follows from the fact that  $\text{Inert}$  is a cocycle: this is proven in detail in [4]. Moreover (ibid.) the Leray index satisfies the following fundamental cochain property: we have

$$m(\ell_\infty, \ell'_\infty) - m(\ell_\infty, \ell''_\infty) + m(\ell'_\infty, \ell''_\infty) = \text{Inert}(\ell, \ell', \ell'') \quad (8)$$

for all triples  $(\ell_\infty, \ell'_\infty, \ell''_\infty)$ . Formula (8) implies in particular that

$$m(\ell_\infty, \ell_\infty) = 0 \quad \text{and} \quad m(\ell_\infty, \ell'_\infty) + m(\ell'_\infty, \ell_\infty) = n - \dim \ell \cap \ell'. \quad (9)$$

One also proves (see again [4, 5]) that  $m$  is in fact the only function  $\Lambda_\infty(n) \times \Lambda_\infty(n) \rightarrow \mathbb{Z}$  satisfying (8) and such that  $m(\ell_\infty, \ell'_\infty)$  remains constant when  $(\ell_\infty, \ell'_\infty)$  moves continuously in such a way that  $\ell$  and  $\ell'$  remain transversal (more generally, if  $\dim \ell \cap \ell'$  does not change). The index  $m$  is a symplectic invariant; more precisely:

$$m(S_\infty \ell_\infty, S_\infty \ell'_\infty) = m(\ell_\infty, \ell'_\infty) \quad \text{for all } S_\infty \in Sp_\infty(n) \quad (10)$$

where  $(S_\infty, \ell_\infty) \mapsto S_\infty \ell_\infty$  is the group action  $Sp_\infty(n) \times \Lambda_\infty(n) \rightarrow \Lambda_\infty(n)$ .

**Example 2** *The Souriau mapping (1) identifies the line  $\ell(\theta) : x \cos \theta + p \sin \theta = 0$  with the complex number  $w = e^{2i\theta}$ ; the Leray index is given by*

$$\begin{aligned} m(\ell(\theta), \ell(\theta')) &= \left\lfloor \frac{\theta - \theta'}{2\pi} \right\rfloor + 1 \quad \text{if } \theta - \theta' \notin \pi\mathbb{Z} \\ m(\ell(\theta), \ell(\theta')) &= 2k \quad \text{if } \theta - \theta' = 2k\pi \quad (k \in \mathbb{Z}) \end{aligned}$$

the covering projection  $\Lambda_\infty(1) \rightarrow \Lambda(1)$  being the mapping  $\theta \mapsto e^{2i\theta}$ .

Recalling that  $\ell_p \equiv 0 \times \mathbb{R}_p^n$  the standard Maslov index  $\mu$  on  $Sp_\infty(n)$  is now defined as follows:

$$\mu(S_\infty) = m(S_\infty \ell_{p,\infty}, \ell_{p,\infty}) \quad (11)$$

where  $\ell_{p,\infty}$  is the homotopy class in  $\Lambda(n)$  of the constant loop through  $\ell_p$ :  $\ell_{p,\infty} \equiv (I, 0)$ . In view of (8) and (10) we have the following essential formula giving the Maslov index of a product:

$$\mu(S_\infty S'_\infty) = \mu(S_\infty) + \mu(S'_\infty) - \text{Inert}(S S'_\infty \ell_p, S \ell_p, \ell_p) \quad (12)$$

for all  $S_\infty, S'_\infty \in Sp_\infty(n)$ . Observe that it immediately follows from (9) that

$$\mu(I_\infty) = 0 \quad (13)$$

where  $I_\infty$  is the unit of  $Sp_\infty(n)$  (i.e., the homotopy class in  $Sp(n)$  of the constant loop through  $I$ ).

**Example 3** Let  $S_t$  be the plane rotation with angle  $t$ , and consider the path  $t \mapsto S_t$ ,  $0 \leq t \leq \alpha$ . The image of the momentum axis  $\ell_p$  by  $S_t$  is the line  $x \cos \alpha + p \sin \alpha = 0$ . It follows from Example 2 that  $\mu(S_{\alpha, \infty}) = [\alpha/2\pi] + 1$  if  $\alpha \neq 2k\pi$  and  $\mu(S_{2k\pi, \infty}) = 2k$ .

The Maslov index of a loop in  $U(n)$  through the identity is *twice* its winding number:

**Proposition 4** Let  $U_\infty$  be the homotopy class in  $Sp(n)$  of a loop  $t \mapsto U_t$  in  $U(n)$  ( $0 \leq t \leq T$ ,  $U_0 = U_T = I$ ). If  $U_\infty \equiv u_\infty = (I, 2k\pi)$  for an integer  $k$  then

$$\mu(U_\infty) = 2k \quad (14)$$

**Proof.** Let us set  $\ell_x = J\ell_p$  (it is just the configuration space  $\mathbb{R}_x^n \times 0$ ), thus  $\ell_x \equiv -I$ . Let  $\ell_{x, \infty} \equiv (-I, n\pi)$ . In view of formula (4) defining the Leray index in the transversal case we have, since  $U\ell_p = \ell_p$ :

$$\begin{aligned} m(U_\infty \ell_{p, \infty}, \ell_{p, \infty}) &= m(U_\infty \ell_{p, \infty}, \ell_{x, \infty}) - m(\ell_{p, \infty}, \ell_{x, \infty}) + \text{Inert}(\ell_p, \ell_p, \ell_x) \\ &= m(U_\infty \ell_{p, \infty}, \ell_{x, \infty}) - m(\ell_{p, \infty}, \ell_{x, \infty}). \end{aligned}$$

Now  $U\ell_p \cap \ell_x = \ell_p \cap \ell_x = 0$  so that formula (3) applies:

$$m(U_\infty \ell_{p, \infty}, \ell_{x, \infty}) = \frac{1}{2\pi}(4k\pi - n\pi + i \text{Tr Log } I) + \frac{n}{2} = 2k.$$

Similarly

$$m(\ell_{p, \infty}, \ell_{x, \infty}) = \frac{1}{2\pi}(0 - n\pi + i \text{Tr Log } I) + \frac{n}{2} = 0$$

hence formula (14). ■

The following generalization of the Maslov index will be used to determine the effect of a change of initial point on the period orbit. One can extend definition (11) by associating to every  $\ell \in \Lambda(n)$  his own private Maslov index  $\mu_\ell$  by the formula

$$\mu_\ell(S_\infty) = m(S_\infty \ell_\infty, \ell_\infty) \quad (15)$$

where  $\ell_\infty$  is the homotopy class in  $\Lambda(n)$  of the constant loop through  $\ell$ . Obviously formula (12) holds for  $\mu_\ell$  as well, replacing  $\ell_p$  by  $\ell$  in the right-hand side. We notice that it follows from (8) that

$$\mu_\ell(S_\infty) - \mu_{\ell'}(S_\infty) = \text{Inert}(S\ell, \ell, \ell') - \text{Inert}(S\ell, S\ell', \ell'). \quad (16)$$

Let  $S_0$  be an arbitrary symplectic matrix and  $S_\infty \in Sp_\infty(n)$ . Then the conjugacy class  $S_0^{-1}S_\infty S_0$  is also an element of  $Sp_\infty(n)$ , and we have:

$$\mu_\ell(S_0^{-1}S_\infty S_0) = \mu_{S_0\ell}(S_\infty). \quad (17)$$

### 3 The Maslov Index of the Monodromy Matrix

Let  $\gamma$  be a periodic orbit of some Hamiltonian  $H \in C^\infty(\mathbb{R}_z^{2n}, \mathbb{R})$ ; we assume for simplicity that the flow  $(f_t^H)$  determined by  $H$  exists for all times. Choose an origin  $z$  on  $\gamma$  so that  $\gamma(t) = f_t^H(z)$ . The Jacobian matrix  $S_t(z) = Df_t^H(z)$  then satisfies the “variational equation”

$$\frac{d}{dt}S_t(z) = JH''(z, t)S_t(z) \quad , \quad S_0(z) = I \quad (18)$$

where  $H''(z, t)$  is the Hessian matrix of  $H$  calculated at  $f_t^H(z)$ . Let  $T$  be the prime period of  $\gamma$ , then  $H''(z, t+T) = H''(z, t)$ . The symplectic matrix  $S_T(z)$  is called the *monodromy matrix*, and it satisfies the relation  $S_{t+T}(z) = S_t(z)S_T(z)$  for all  $t$ . We denote by  $S_{T,\infty}$  the homotopy class in  $Sp(n)$  of the symplectic path  $t \mapsto S_t$ ,  $0 \leq t \leq T$ . We begin by making a pedestrian remark: since  $S_T^r = S_{rT}$  for every integer  $r \in \mathbb{Z}$  formula (12) allows us at once to calculate the Maslov index for repetitions of the prime periodic orbit. For instance, if  $r = 2$ ,

$$\mu(S_{2T,\infty}) = 2\mu(S_{T,\infty}) - \text{Inert}(S_T^2\ell_p, S_T\ell_p, \ell_p). \quad (19)$$

We will actually use this formula with profit to determine the Maslov index  $\mu(S_{T,\infty})$  itself. Let us namely make the following observation. When one writes the monodromy matrix  $S_T$  in exponential form  $S_T = e^{TX}$  it is not true in general that the matrix  $X$  is in the symplectic Lie algebra; in fact  $X$  is usually not even real! (This is due to the presence of inverse hyperbolic blocks when  $S_T$  is put in normal form.) Since  $S_{t+2T}(z) = S_t(z)(S_T(z))^2$  we can write

$$S_t(z) = P_t(z)e^{tX(z)} \quad (20)$$

where  $t \mapsto P_t(z)$  is  $2T$ -periodic and  $X(z)$  is real; we will assume that  $X(z) \in \mathfrak{sp}(n)$  so that both  $P_t(z)$  and  $e^{tX(z)}$  are symplectic. For notational simplicity we drop for the moment being any reference to the origin  $z$  of the periodic orbit;  $z$  will be reinstated when we set out to discuss the effect of change of origin on the Maslov index.

Let  $S_{T,\infty}$  (resp.  $S_{2T,\infty}$ ) the homotopy class of the symplectic path  $t \mapsto S_t$ ,  $0 \leq t \leq T$  (resp.  $0 \leq t \leq 2T$ ). Both  $S_{T,\infty}$  and  $S_{2T,\infty}$  are elements of  $Sp_\infty(n)$ , and  $S_{2T,\infty} = S_{T,\infty}^2$ .

**Theorem 5** (\*) *The Maslov index  $\mu(S_{T,\infty})$  of the symplectic path  $t \mapsto S_t$ ,  $0 \leq t \leq T$ , is given by the formula*

$$\mu(S_{T,\infty}) = \frac{1}{2}(\mu(P_{2T,\infty}) + \mu(e_\infty^{2TX}) + \text{Inert}(S_{2T}\ell_p, S_T\ell_p, \ell_p)) \quad (21)$$

where  $P_{2T,\infty}$  (resp.  $e_\infty^{2TX}$ ) is the homotopy class in  $Sp(n)$  of the path  $t \mapsto P_t$  (resp.  $t \mapsto e^{tX}$ ),  $0 \leq t \leq 2T$ . (\*\*) Let  $P_t = U_t e^{Y_t}$  be the polar decomposition of  $P_t$ , that is  $U_t \in U(n)$  and  $Y_t$  is a symmetric matrix in  $\mathfrak{sp}(n)$ . Then

$$\mu(P_{2T,\infty}) = \mu(U_{2T,\infty}) = 2k \quad (22)$$

where  $U_{2T,\infty}$  is the homotopy class in  $Sp(n)$  (or in  $U(n)$ ) of  $t \mapsto U_t$  and  $k$  is the winding number defined by (14). (\*\*\*) If the monodromy matrix is  $S_T = e^{TX}$  then its Maslov index is given by

$$\mu(S_{T,\infty}) = \mu(e_\infty^{TX}) + k \quad (23)$$

where  $e_\infty^{TX}$  is the homotopy class of  $t \mapsto e^{tX}$ ,  $0 \leq t \leq 2T$ .

**Proof.** (\*) By definition of the group structure of  $Sp_\infty(n)$  we have  $S_{2T,\infty} = P_{2T,\infty} e_\infty^{2TX}$ . Using (12), the fact that  $P_{2T} = I$ , and the definition of the index of inertia, we have

$$\begin{aligned} \mu(S_{2T,\infty}) &= \mu(P_{2T,\infty}) + \mu(e_\infty^{2TX}) - \text{Inert}(P_{2T} e^{2TX} \ell_p, e^{2TX} \ell_p, \ell_p) \\ &= \mu(P_{2T,\infty}) + \mu(e_\infty^{2TX}) - \text{Inert}(e^{2TX} \ell_p, e^{2TX} \ell_p, \ell_p) \\ &= \mu(P_{2T,\infty}) + \mu(e_\infty^{2TX}) \end{aligned}$$

and formula (21) follows in view of (19). (\*\*) To prove (22) we begin by noting that in view of the uniqueness of the symplectic polar decomposition we have both  $U_0 = U_{2T} = I$  and  $e^{Y_0} = e^{Y_{2T}} = I$ . Writing  $P_{2T,\infty} = U_{2T,\infty} e_\infty^{Y_{2T}}$  we have, again by (12) and the definition of the index of inertia:

$$\begin{aligned} \mu(P_{2T,\infty}) &= \mu(U_{2T,\infty}) + \mu(e_\infty^{Y_{2T}}) - \text{Inert}(P_{2T} \ell_p, e^{Y_{2T}} \ell_p, \ell_p) \\ &= \mu(U_{2T,\infty}) + \mu(e_\infty^{Y_{2T}}) \end{aligned}$$

because  $P_{2T} \ell_p = e^{Y_{2T}} \ell_p = \ell_p$  implies that

$$\text{Inert}(P_{2T} \ell_p, e^{Y_{2T}} \ell_p, \ell_p) = \text{Inert}(\ell_p, \ell_p, \ell_p) = 0.$$

Formula (22) will follow if we show that  $\mu(e_\infty^{Y_{2T}}) = 0$ . Now,  $e_\infty^{Y_{2T}}$  is the homotopy class in  $Sp(n)$  of the loop  $t \mapsto e^{Y_t}$ ,  $0 \leq t \leq T$ . The subset of  $Sp(n)$  consisting of positive definite matrices is simply connected, hence that loop is contractible to a point, and thus homotopic to the identity. The relation  $\mu(e_\infty^{Y_{2T}}) = 0$  follows in view of (13). (\*\*\*) By definition of the product in  $Sp_\infty(n)$  we have  $e_\infty^{2TX} = e_\infty^{TX} e_\infty^{TX}$  and hence

$$\mu(e_\infty^{2TX}) = 2\mu(e_\infty^{TX}) - \text{Inert}(e^{2TX} \ell_p, e^{TX} \ell_p, \ell_p)$$

in view of (12). It follows from (21) and (22) that

$$\begin{aligned} \mu(S_{T,\infty}) &= \mu(e_\infty^{TX}) + k + \\ &\quad \frac{1}{2}(\text{Inert}(S_{2T} \ell_p, S_T \ell_p, \ell_p) - \text{Inert}(e^{2TX} \ell_p, e^{TX} \ell_p, \ell_p)) \end{aligned}$$

hence formula (23) since  $e^{2TX} = S_{2T}$  and we are assuming that  $e^{TX} = S_T$ . ■

There remains to investigate what happens to the Maslov index when we change the initial point  $z$ . Let us consider two solutions  $t \mapsto S_t(z)$  and  $t \mapsto S_t(z')$  of the variational equation (18) where  $z$  and  $z'$  are two points on the

same periodic orbit  $\gamma$  of the Hamiltonian  $H$ . We begin with the following preliminary remark: since  $z$  and  $z'$  are on the same orbit there exists a time  $t'$  such that  $z = f_{t'}^H(z')$  and hence, using the chain rule together with the relation  $f_{t+t'}^H = f_t^H f_{t'}^H$ :

$$S_{t+t'}(z') = S_t(z)S_{t'}(z') \quad (24)$$

(recall that  $S_t(z)$  is the Jacobian matrix of  $f_t^H$  calculated at  $z$ ).

**Theorem 6** (\*) *The Maslov indices  $\mu(S_{T,\infty}(z))$  and  $\mu(S_{T,\infty}(z'))$  corresponding to two points  $z$  and  $z'$  lying on the same periodic orbit of the Hamiltonian  $H$  are related by the formula:*

$$\mu(S_{T,\infty}(z')) = \mu_{S'\ell_p}(S_{T,\infty}(z)) \quad \text{with } S' = S_{t'}(z'). \quad (25)$$

Equivalently, in view of (16):

$$\mu(S_{T,\infty}(z)) - \mu(S_{T,\infty}(z')) = \text{Inert}(S_T\ell_p, \ell_p, S'\ell_p) - \text{Inert}(S_T\ell_p, S_T S'\ell_p, S'\ell_p). \quad (26)$$

(\*\*) *The winding number  $\mu(U_{2T,\infty})$  in (22) is not affected by the change of  $z$  in  $z'$ :*

$$\mu(U_{2T,\infty}(z)) = \mu(U_{2T,\infty}(z')). \quad (27)$$

**Proof.** (\*) Formulas (25) and (26) are equivalent in view of (16). It follows from (24) that we have  $S_{t'+T}(z') = S_T(z)S'$ ; since on the other hand  $S_{t'+T}(z') = S_{t'}(z')S_T(z')$  by the properties of the monodromy matrix, we thus have the following conjugacy relation between the monodromy matrices at different points of the orbit:

$$S_T(z') = (S')^{-1}S_T(z)S'. \quad (28)$$

We next note that the two symplectic paths

$$\Sigma : t \mapsto S_t(z') \quad \text{and} \quad \Sigma' : t \mapsto (S')^{-1}S_t(z)S' \quad (0 \leq t \leq T)$$

are homotopic with fixed endpoints (that they are homotopic is clear letting  $t' \rightarrow 0$ ; they moreover have same endpoints  $I$  and  $S_T(z')$  in view of (28)). These paths thus have same Maslov index  $\mu(S_{T,\infty}(z'))$ , and hence

$$\mu(S_{T,\infty}(z')) = \mu((S')^{-1}S_{T,\infty}(z)S') = \mu_{S'\ell_p}(S_{T,\infty}(z))$$

proving (25). (\*\*) We have, in view of (16), and using the fact that  $U_T = I$ :

$$\begin{aligned} \mu(U_{2T,\infty}(z)) - \mu(U_{2T,\infty}(z')) &= \text{Inert}(U_{2T}\ell_p, \ell_p, S'\ell_p) - \text{Inert}(U_{2T}\ell_p, U_T S'\ell_p, S'\ell_p) \\ &= \text{Inert}(\ell_p, \ell_p, S'\ell_p) - \text{Inert}(\ell_p, S'\ell_p, S'\ell_p) \\ &= 0. \end{aligned}$$

■

## 4 Relation With Gutzwiller's Index

Let us shortly explain in which way the Maslov index we have constructed is related to the index appearing in Gutzwiller's trace formula

$$\delta g(E) = \frac{1}{\pi \hbar} \sum_{\gamma} \frac{T_{\gamma}}{\sqrt{|\det(\tilde{M}_{\gamma} - I)|}} \cos\left(\frac{1}{\hbar} \mathcal{A}_{\gamma} - \frac{\pi}{2} \xi_{\gamma}\right)$$

for the oscillating part of the level density of a system which has only isolated periodic orbits  $\gamma$  with prime periods  $T_{\gamma}$ . In the formula above  $\tilde{M}_{\gamma}$  is the stability matrix of the periodic orbit  $\gamma$ , and  $\mathcal{A}_{\gamma}$  the action of that orbit; the term  $\frac{\pi}{2} \xi_{\gamma}$  can be viewed as giving the argument of the square root  $\sqrt{|\det(\tilde{M}_{\gamma} - I)|}$ . One shows [2, 3, 11] that  $\xi_{\gamma}$  can be written as a sum

$$\xi_{\gamma} = \mu_{\gamma} + \nu_{\gamma}$$

of two contributions:  $\mu_{\gamma}$  is the phase index appearing in the semi-classical time-dependent Green function  $G(x, x'; t)$  while  $\nu_{\gamma}$  arises when taking the trace of  $G$  by stationary phase integrations transverse to  $\gamma$  (see [8] for a detailed account of these procedures); it is shown in [3] that  $\xi_{\gamma}$  does not depend on the choice of the initial point on the orbit  $\gamma$  while the individual terms  $\mu_{\gamma}$ ,  $\nu_{\gamma}$  are in general dependent of that point. We can give the following (tentative) interpretation of the relation between our constructs and the Gutzwiller index  $\xi_{\gamma}$ . Assume that the orbit  $\gamma$  lies on an invariant Lagrangian submanifold  $\mathbb{V}$  of phase space. To each point  $\gamma(t)$  of that orbit we can associate a Lagrangian plane  $\ell(t)$ , namely the tangent space to  $\mathbb{V}$  at  $\gamma(t)$ . We thus obtain a loop  $\lambda$  in the Lagrangian Grassmannian  $\Lambda(n)$ . Now there is a fundamental relation between the Leray index and the usual Arnol'd–Maslov index  $\text{Mas}(\lambda)$  of the loop  $\lambda$ : we have, for every  $\ell_{\infty} \in \Lambda_{\infty}(n)$ ,

$$\text{Mas}(\lambda) = m(\ell_{\infty}(T), \ell_{\infty}) - m(\ell_{\infty}(0), \ell_{\infty}) \quad (29)$$

where  $\ell_{\infty}(0)$  is the homotopy class in  $\Lambda(n)$  of the origin  $\ell(0) = \lambda(0)$  of  $\lambda$  and  $\ell_{\infty}(T)$  that of the whole loop (see [4, 5, 9]). We now observe that since  $\gamma(t)$  is obtained from  $\gamma(0)$  by  $f_t^H$  it follows that  $\lambda(t)$  is obtained from  $\lambda(0)$  by  $S_t$ ; choosing  $\ell_{\infty} = \ell_{\infty}(0)$  in (29) and using the first equality (9), we get

$$\text{Mas}(\lambda) = m(S_{T,\infty} \ell_{\infty}(0), \ell_{\infty}(0)) = \mu_{\ell(0)}(S_{T,\infty}).$$

If—as suggested in [3] and [11]—we have  $\xi_{\gamma} = \text{Mas}(\lambda)$ , it follows that we actually have

$$\xi_{\gamma} = \mu_{\ell(0)}(S_{T,\infty}). \quad (30)$$

Note that since  $\text{Mas}(\lambda)$  is independent of the choice of origin so is the case for  $\mu_{\ell(0)}(S_{T,\infty})$  (this property also easily follows from (17) and (28), alternatively (26)).

We will come back to a detailed analysis of the relation between the Gutzwiller and Leray indices (and in particular of the validity of (30)) in a forthcoming paper.

## References

- [1] BRACK, M AND JAIN, S R 1995 Phys. Rev. A **51**, 3462
- [2] CREAGH, S C AND LITTLEJOHN, R G 1991 Phys. Rev. A **44**, 836
- [3] CREAGH, S C , ROBBINS J M AND LITTLEJOHN, R G 1990 Phys. Rev. A **42**, 1907
- [4] DE GOSSON, M 1992 J. Math. Pures et Appl. **71**, 429; announced in: 1990 C.R. Acad. Sci., Paris, **309**, Série I, 279; 1990 C.R. Acad. Sci., Paris, **310**, Série I, 245
- [5] DE GOSSON, M 1997 *Maslov Classes, Metaplectic Representation and Lagrangian Quantization*, Research Notes in Mathematics **95** (Wiley–VCH, Berlin)
- [6] DE GOSSON, M 2002 J. Phys. A: Math. Gen. **35**, 6825
- [7] GUILLEMIN, V, AND STERNBERG, S 1978 *Geometric Asymptotics*, Math. Surveys Monographs **14** (Amer. Math. Soc., Providence R.I.)
- [8] GUTZWILLER, M C 1990 *Chaos in Classical and Quantum Mechanics*, Interdisciplinary Applied Mathematics (Springer-Verlag).
- [9] LERAY, J 1981 *Lagrangian Analysis and Quantum Mechanics, a mathematical structure related to asymptotic expansions and the Maslov index* (the MIT Press, Cambridge, Mass.)
- [10] PLETYUKHOV, M AND BRACK, M 2003 J. Phys. A: Math. Gen. **36**, 9449
- [11] ROBBINS, J M 1991 Nonlinearity **4**, 343
- [12] SUGITA, A. 2001 Annals of Physics **288**, 277